Auctions with Affiliated Information

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Workshop on Mechanism Design

I.S.I. Delhi

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Questions

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- **Revenue comparisons**, and

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- **Efficiency**

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- **Efficiency**

in common types of auctions when bidder information is correlated

Common auctions for a single object

Open Format

Sealed-Bid Format

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Sealed-Bid Format

Dutch or Descending-Price

Common auctions for a single object

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Common auctions for a single object

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Sealed-Bid Format

First-Price

Second-Price

Equivalences between auctions for a single object

Open	**Sealed-Bid**	**Equivalence**
Dutch or Descending-Price	First-Price	*Always*

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Dutch or Descending-Price	First-Price	*Always*
English or Ascending-Price	Second-Price	*Independent information or two bidders*

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  $$f(v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_n) \;=\; f(v_{i_1}, v_{i_2}, \ldots, v_{i_n}, x_{i_1}, x_{i_2}, \ldots, x_{i_n})$$

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- Seller's cost is 0. Bidders' valuation $0 \leq V_i \leq \overline{V}$
- All this is common knowledge

Symmetric, Interdependent Values Model

- Bidder i's expected valuation is a function of signals $X_1, X_2, \ldots, X_n$

$$\begin{aligned} v(x_i, x_{-i}) &= \mathrm{E}[V_i | X_i = x_i, X_{-i} = x_{-i}] \\ &= \mathrm{E}[V_j | X_j = x_i, X_{-j} = x_{-i}] \end{aligned}$$

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- Symmetry implies that permutations within x_{-i} do not change $v(\cdot)$.

  For example,

  $$v(x_i, x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = v(x_i, x_2, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

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- **Interdependent values, independent information:**

  X_i, X_j independent.

  For example, X_i are i.i.d. $U[0,1]$ and $V_i = X_i + c\sum_{j\neq i} X_j$

Cases of interest

- **Interdependent values:** $v(x_i, x_{-i})$
- **Pure common values:** $V_1 = V_2 = \ldots = V_n$
- **Private values:** $V_i = X_i$
- ~~**Private independent values:** X_i, X_j independent~~
- ~~**Interdependent values, independent information:**~~

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Affiliation

$\mathbf{Z} = (Z_1, Z_2, \ldots, Z_m)$ are random variables
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The random variables $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_m)$ are *affiliated* if for all $\mathbf{z}$, $\mathbf{z}'$

$$f(\mathbf{z} \vee \mathbf{z}')f(\mathbf{z} \wedge \mathbf{z}') \geq f(\mathbf{z})f(\mathbf{z}')$$

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If random variables $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_m)$ are affiliated then

A1. Any subset of random variables $(Z_1, Z_2, \ldots, Z_m)$ are affiliated.

A2. Z_1 and the order statistics of $(Z_2, \ldots, Z_m)$ are affiliated.

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A3. With Y_1 equal to the largest of $Z_2, \ldots, Z_m$

$$\frac{g_{Y_1|Z_1}(y|z')}{G_{Y_1|Z_1}(y|z')} \leq \frac{g_{Y_1|Z_1}(y|z)}{G_{Y_1|Z_1}(y|z)}, \qquad \forall y,\ \forall z' < z$$

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A4. If $h(z_1, z_2, \ldots, z_m)$ is an increasing function then

$$\mathrm{E}[h(z_1, z_2, \ldots, z_m) \,|\, (z_1^a, z_2^a, \ldots, z_m^a) \leq \mathbf{Z} \leq (z_1^b, z_2^b, \ldots, z_m^b)]$$

is increasing in each z_i^a, z_i^b.

Assumption

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$$\begin{aligned} v(x_1, x_2, \ldots, x_n) &= E[V_1 | X_1 = x_1, X_1 = x_2, \ldots, X = x_n] \\ \text{and} \qquad w(x, y) &\equiv E[V_1 | X_1 = x, Y_1 = y] \end{aligned}$$

$v(\cdot)$ and $w(\cdot)$ are increasing functions.

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Further,

$$\frac{g(y|x')}{G(y|x')} \leq \frac{g(y|x)}{G(y|x)}, \qquad \forall y,\ \forall x' < x$$

where g is conditional density & G the conditional cdf of Y_1 given X_1.

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Claim: $b_s(x) \equiv w(x, x)$ is a symmetric Nash equilibrium strategy.

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If bidder 1 wins the auction, he pays $b_s(y) = w(y,y)$. Because

$$w(x,y) - w(y,y) \ \lesseqgtr\ 0 \qquad \text{as} \qquad x \ \lesseqgtr\ y$$

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In fact, each bidder playing b_s constitutes an ex post equilibrium.

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Winner's curse is **not** an equilibrium phenomenon

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n	1	2	5	10
$E[\max \epsilon_i] = E[\max(X_i - V)]$	0	0.564σ	1.163σ	1.539σ

Winner's curse in oil lease auctions

Bids on offshore oil tracts ($ millions), 1967-69

	Louisiana	Santa Barbara	Texas	Alaska
Highest bid	32.5	43.5	43.5	10.5
2nd highest bid	17.7	32.1	15.5	5.2
Lowest bid	3.1	6.1	0.4	0.4
Money left on table	14.8	11.4	28	5.3
Highest/Lowest ratio	10	7	109	26

From Capen, Clapp, and Campbell, "Competitive Bidding in High Risk Situations," *Journal of Petroleum Technology,* 1971, 23, 641-653.

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$$\begin{aligned} b_f(x) &\equiv \int_0^x w(y,y)\,dL(y|x) \\ \text{where} \qquad L(y|x) &= \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)}dt\right) \end{aligned}$$

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and $g(y|x)$ is the density and $G(y|x)$ is the cdf of $Y_1 = y$ given $X_1 = x$.

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$b_f(x)$ is the solution to the differential equation

$$\frac{db(x)}{dx} = [w(x,x) - b(x)]\frac{g(x|x)}{G(x|x)}$$

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Proof: Bidder 1's expected profit when $X_1 = x$ and he bids as if $X_1 = \hat{x}$ is

$$\Pi(\hat{x}, x) = \int_0^{\hat{x}} w(x,y) g(y|x) dy - b_f(\hat{x}) G(\hat{x}|x)$$

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\frac{\partial \Pi}{\partial \hat{x}} &= \left\{ [w(x, \hat{x}) - b_f(\hat{x})] \frac{g(\hat{x}|x)}{G(\hat{x}|x)} - \frac{db_f(\hat{x})}{d\hat{x}} \right\} G(\hat{x}|x)
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If $\hat{x} > x$ then $\frac{g(\hat{x}|x)}{G(\hat{x}|x)} \leq \frac{g(\hat{x}|\hat{x})}{G(\hat{x}|\hat{x})}$ and $w(x, \hat{x}) \leq w(\hat{x}, \hat{x})$.

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Similarly, if $\hat{x} < x$ then $\frac{\partial \Pi}{\partial \hat{x}} \geq 0$. □

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Claim: Second-price auction yields greater expected revenue than first-price.

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\end{aligned}$$

□

Importance of affiliated information signals

An example with two bidders:

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Revenue equivalence, even though V_1, V_2 are affiliated!

English Auction with 3 bidders

Define

$$\begin{aligned} b_{e,0}(x) &= \mathrm{E}[V_1 | X_1 = x, X_2 = x, X_3 = x] \\ b_{e,1}(x; p) &= \mathrm{E}[V_1 | X_1 = x, X_2 = x, X_3 = b_{e,0}^{-1}(p)] \end{aligned}$$

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Therefore, bidder 1 maximizes surplus by playing $(b_{e,0}, b_{e,1})$. □

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$$\begin{aligned} \Longrightarrow \qquad P_s &= \mathrm{E}[\mathrm{E}[w(Y_1, Y_1) | X_1, X_1 > Y_1]] \\ &\leq \mathrm{E}[\mathrm{E}[v(\max\{X_2, X_3\}, X_2, X_3) | X_1, X_1 > Y_1]] \;=\; P_e \qquad \square \end{aligned}$$

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This leads to more aggressive bidding and, as the pie is fixed in all three auctions, greater expected revenues for the auctioneer.

Other implications of the Linkage Principle

Honesty is the best policy for the auctioneer.

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Greater revenues with royalty payments.

Caveats to the Linkage Principle

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Is that efficient?

An example of inefficient allocation

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Therefore, the bidder with the lower valuation obtains object!

A sufficient condition for efficiency

Recall that, for our symmetric model,

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Single-crossing condition: If

$$\frac{\partial v(x_1, x_2, \ldots, x_n)}{\partial x_1} \geq \frac{\partial v(x_1, x_2, \ldots, x_n)}{\partial x_2}$$

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In asymmetric models, English auctions are more efficient than second-price auctions are more efficient than first-price auctions.